Limiting distribution of the geometric DLPP

Yuxuan Zong

USTC

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Overview

1. Introduction and main results
2. Distribution of $T(m, n)$ in the geometric DLPP
3. Limiting distribution of $T(m, n)$ in the geometric DLPP
4. Limiting distribution of $T(m, n)$ in the exponential DLPP

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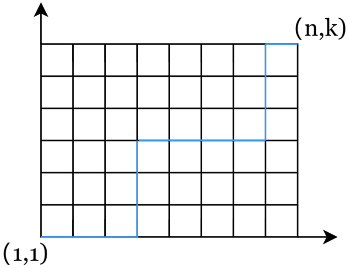
Directed last passaage percolation(DLPP)



DLPP

- i.i.d. random variables w_{ij} at each site $(i,j) \in \mathbb{Z}^2$. Let $\mu = \mathbb{E}[w_{ij}], \mathrm{Var}(w_{ij}) = \sigma^2$.
- Admissible paths: $\Pi(n,k)$= the set of "up/right" paths from $(1,1)$ to (n,k).
- Last passage time: $T(n,k) = \max_{\pi \in \Pi(n,k)} \left\{ \sum_{(i,j)\in\pi} w_{ij} \right\}$.
- Related models: Corner growth, TASEP, $\cdots$

Some expectation consequences of these integrable models:

$$\lim_{n\to+\infty} \frac{\mathbb{E}[T([\gamma n], n)]}{n} = a(\gamma). \tag{1}$$

Theorem (Rost, 1981)

For $w_{ij} \sim Exp(1)$,

$$a(\gamma) = (1+\sqrt{\gamma})^2$$

Theorem (Johansson, 2000)

For $\mathbb{P}(w_{ij} = k) = (1-q)q^k, \quad k \in \mathbb{N}, \quad 0 < q < 1,$

$$a(\gamma) = \frac{q(1+\gamma) + 2\sqrt{q\gamma}}{1-q}.$$

Theorem (Johansson, 2000)

For geometric, exponential DLPP, we have that

$$\frac{T([\gamma n], n) - a(\gamma)n}{b(\gamma)n^{1/3}} \xrightarrow{D} F_{TW}, \quad n \to +\infty.$$

- For $\mathbb{P}(w_{ij} = k) = (1-q)q^k, \quad k \in \mathbb{N}, \quad 0 < q < 1,$

$$b(\gamma) = \frac{q^{1/6}}{(1-q)x^{1/6}}(\sqrt{\gamma} + \sqrt{q})^{2/3}(1 + \sqrt{q\gamma})^{2/3}.$$

- $Exp(1) : b(\gamma) = \dfrac{(1+\sqrt{\gamma})^{4/3}}{\gamma^{1/6}};$

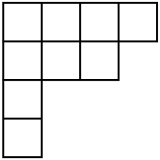
Overview

Definition

A **partition** is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of non-increasing integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is finite. The number of non-zero numbers λ_i is called the length of λ.
A partition can be depicted by **Young diagram**. These are arrays of left justified unit boxes, the *i* th row of which has λ_i boxes.
The set of all partitions is denoted by $\mathbb{Y}$. The set of all partitions of length at most N is denoted by $\mathbb{Y}_N$.

Example

The partition $\lambda = (4, 3, 1, 1\cdots)$ has the Young diagram with $|\lambda| = 9$. We also have $\lambda \in \mathbb{Y}_4 \subset \mathbb{Y}_5 \subset \cdots$



Schur plynoimials

Definition

To each $\lambda \in \mathbb{Y}_N$, we associate the Schur polynomial defined by

$$s_\lambda(x_1, \cdots, x_N) = \frac{\det[x_j^{\lambda_i+N-i}]_{i,j=1}^N}{\det[x_j^{N-i}]_{i,j=1}^N} = \frac{\det[x_j^{\lambda_i+N-i}]_{i,j=1}^N}{\prod\limits_{1\le i<j\le N}(x_i - x_j)}$$

Theorem (Combinatorial definition of Schur polynomials)

For $\lambda \in \mathbb{Y}_N$,

$$s_\lambda(x_1, \cdots, x_N) = \sum_{\emptyset=\lambda^{(0)}\preceq\lambda^{(1)}\preceq\cdots\preceq\lambda^{(N)}=\lambda} \prod_{k=1}^N x_k^{|\lambda^{(k)}|-|\lambda^{(k-1)}|}$$

where $\lambda^{(k)} \in \mathbb{Y}_k$.

For $\lambda \in \mathbb{Y}_N$ and $\mu \in \mathbb{Y}_{N-1}$, we say that λ and μ interlace, denoted by $\mu \preceq \lambda$, if $\lambda_{i+1} \le \mu_i \le \lambda_i$ for all $1 \le i \le N-1$.

Definition

A semistandard Young tableau of shape λ and rank N is a filling of the boxes of the Young diagram λ with numbers $1, \cdots, N$ such that the numbers along a row increase weakly and the numbers along a column increase strictly. Let $d_\lambda(N)$ is the number of semistandard Young tableaux of shape λ and rank N.

Proposition

For $\lambda \in \mathbb{Y}_N$, we have that

$$d_\lambda(N) = s_\lambda(\underbrace{1, \cdots, 1}_{N}) = \prod_{1 \le i < j \le N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Definition

Let $\mathbb{M}_k(M,N)$ be the set of $M\times N$ matrices $A=(A_{ij})$ with $A_{ij}\in\mathbb{Z}_+$ and $\sum_{i,j}A_{i,j}=k$. Let $\mathbb{M}(M,N)=\bigcup_k \mathbb{M}_{k=0}^{\infty}(M,N)$

Theorem (RSK: Robinson-Schensted-Knuth correspondence)

There is a bijection, denoted by RSK, between $\mathbb{M}_k(M,N)$ and the pair of semistandard Young tableaux (P,Q) such that

- P and Q have the same shape λ with $|\lambda|=k$.
- P has rank M.
- Q has rank N.

Furthermore,

$$\lambda_1 = G(A) := \max_{\pi} \sum_{(i,j)\in\pi} A_{ij}$$

where the maximum is taken over all up/right "path" from $(1,1)$ to (M,N), $\pi=\left\{(i_k,j_k)_{k=1}^{M+N-1}\right\}$ satisfying

$$(i_1,j_1)=(0,0),(i_{M+N1},j_{M+N1})=(M,N),(i_{k+1},j_{k+1})(i_k,j_k)\in\{(1,0),(0,1)\}.$$

Example

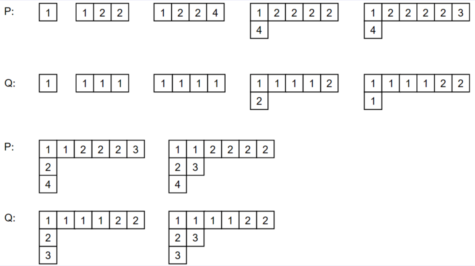
We consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $G(A) = 6$, and A is equivalent to the two line array

$$\begin{matrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 4 & 2 & 2 & 3 & 1 & 2 \end{matrix}$$

The matrix A is mapped to the semistandard Young tableaux (P, Q):



Theorem

For any $m \geq n \geq 1$,

$$\mathbb{P}(T(m,n) \leq x) = \frac{1}{Z_{m,n}} \sum_{0 \leq h_i \leq x+n-1} \Delta_n(h)^2 \prod_{i=1}^{n} \binom{h_i + m - n}{h_i} q^{h_i} \tag{2}$$

where $Z_{m,n}$ is the normalization constant

$$Z_{m,n} = \sum_{h \in \mathbb{Z}_+^n} \Delta_n(h)^2 \prod_{i=1}^{n} \binom{h_i + m - n}{h_i} q^{h_i} \tag{3}$$

and $\Delta_n(h)$ is a Vandermonde determinant

$$\Delta_n(h) = \prod_{1 \leq i < j \leq n} |h_i - h_j|.$$

Proof idea: Matrix A in RSK $\longleftrightarrow$ Young tableaux(P,Q) $\longleftrightarrow$ $s_\lambda(1,\cdots,1)$.

Overview

Meixner polynomials

Let $K = m - n + 1, m = [\gamma n]$. Define the measure μ^K on $\mathbb{Z}_+$ by

$$\mu^K(x) = \binom{x+K-1}{x} q^x, \quad x \in \mathbb{Z}_+.$$

Rewrite (2) as

$$\mathbb{P}(T(m,n) \le x) = \frac{1}{Z_{m,n}} \sum_{0 \le h_i \le x+n-1} \Delta_n(h)^2 \prod_{i=1}^{n} \mu^K(h_i) \tag{4}$$

Define the probability measure $\mathcal{Q}_{m,n}$ on $\mathbb{Z}_+^n$ by

$$\mathcal{Q}_{m,n}(A) = \frac{1}{Z_{m,n}} \sum_{h \in \mathbb{Z}_+^n : h \in A} \Delta_n(h)^2 \prod_{i=1}^{n} \mu^K(h_i) \tag{5}$$

Then

$$\begin{aligned}\mathbb{P}(T(m,n) \le x) &= \int_{\mathbb{R}^n} \prod_{j=1}^{n} (1 - \mathbf{1}_{[x+n,\infty)}(h_j)) \mathcal{Q}_{m,n}(\mathrm{d}\vec{h}) \\ &= 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{h \in \mathbb{Z}_+^k, h_i \ge n+x} \det[K_n(h_i,h_j)]_{i,j=1}^{n}.\end{aligned}$$

Meixner polynomials

Definition

Let$\left\{M_j^K(x) : j \in \mathbb{Z}_+\right\}$ denote the polynomials that are orthonormal under the weights $\mu^K(x)$:

$$\sum_{x \in \mathbb{Z}_+} M_i^K(x) M_j^K(x) \mu^K(x) = \delta_{i,j}, \quad i, j \in \mathbb{Z}_+$$

and have positive leading coefficient $\kappa_i > 0 : M_i^K(x) = \kappa_i x^i + \cdots$ which called Meixner polynomials.

Define Meixner kernel

$$K_n(x, y) = \sum_{i=0}^{n-1} M_i^K(x) M_i^K(y) \mu^K(x)^{1/2} \mu^K(y)^{1/2}. \tag{6}$$

The polynomials $M_j^K(x)$ are multiples of the standard Meixner polynomials

$$M_n^K(x) = \frac{(-1)^n}{d_n} m_n^K(x), \tag{7}$$

where

$$d_n^2 = \frac{n!(n+K-1)!}{(1-q)^K q^n (K-1)!}. \tag{8}$$

Proposition

For $x \in \mathbb{R}$ and $n \in \mathbb{Z}_+$ we have the formula

$$m_n^K(x) = (-1)^n n! \sum_{k=0}^{n} \binom{x}{k} \binom{-x-K}{n-k} q^{-k} \tag{9}$$

and the leading coefficient in $\kappa_n^K(x)$ is $(\frac{q-1}{q})^n$ and consequently

$$\kappa_n = \frac{1}{d_n} \left(\frac{1-q}{q} \right)^n . \tag{10}$$

The generating function is

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} m_n^K(x) = \left(1 - \frac{z}{q}\right)^x (1-z)^{-x-K} \tag{11}$$

for $x \in \mathbb{R}$ and complex z such that $|z| < q$.

Christoffel-Darboux formula

By the Christoffel-Darboux formula , for $x \neq y$ in $\mathbb{Z}_+$

$$K_n(x,y) = \frac{\kappa_{n-1}}{\kappa_n} \cdot \frac{M_n^K(x)M_{n-1}^K(y) - M_n^K(y)M_{n-1}^K(x)}{x-y} \mu^K(x)^{1/2}\mu^K(y)^{1/2} \quad (12)$$

$$= \frac{-q}{(1-q)d_{n-1}^2} \cdot \frac{m_n^K(x)m_{n-1}^K(y) - m_n^K(y)m_{n-1}^K(x)}{x-y} \mu^K(x)^{1/2}\mu^K(y)^{1/2}, \quad (13)$$

Fredholm determinant

$$\mathbb{P}(T(m,n) \le x) = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{h \in \mathbb{Z}_+^k, h_i \ge n+x} \det[K_n(h_i, h_j)]_{i,j=1}^{n} \tag{14}$$

Let $b \ge 0$ be a constant and assume that $\rho_n \to \infty$ as $n \to \infty$ along positive integers and assume $\rho_n = o(n)$. Suppose furthermore that $K_n : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $n \ge 1$, satisfies the following properties:

Lemma

(i) Let $M_1 > 0$ be a given constant. For all $n \geq 1, \tau \geq -M_1$, there is a constant C such that

$$\sum_{m=1}^{\infty} K_n(bn + \rho_n\tau + m, bn + \rho_n\tau + m) \leq C. \tag{15}$$

(ii) Given $\varepsilon > 0$, there is an $L > 0$ so that

$$\sum_{m=1}^{\infty} K_n(bn + \rho_n L + m, bn + \rho_n L + m) \leq \varepsilon \tag{16}$$

for all $n \geq 1$.

(iii) Let $M_0 > 0$ be a given constant. If $\mathbb{A}(\xi, \eta)$ is the Airy kernel defined by (**??**), then

$$\lim_{n\to\infty} \rho_n K_n(bn + \rho_n\xi, bn + \rho_n\eta) = \mathbb{A}(\xi, \eta) \tag{17}$$

uniformly for $\xi, \eta \in [-M_0, M_0]$.

(iv) The matrix $(K_n(x_i, x_j))_{i,j=1}^{n}$ is positive definite for any $x_i, x_j \in [0, \infty), k \geq 1$.

Then, for each fixed $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \sum_{h\in\mathbb{N}^k} \det[K_n(bn + \rho_n t + h_i, bn + \rho_n t + h_j)]_{i,j=1}^{k} = F_{TW}(t), \tag{18}$$

Lemma

The Meixner kernel satisfies the properties (i) to (iv) in the above Lemma with $b = a(\gamma)$ and $\rho_n = b(\gamma)n^{1/3}$, where $a(\gamma), b(\gamma)$ are given by Theorem.

Combining (14) and (18) , we obtain the result.

Overview

From geometric r.v. to exponential r.v.

Lemma

If random variables $\{X_n\}$ are geometric with parameter $1/n$, then the distribution of X_n/n converges to $Exp(1)$.

Proof.

Note that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n} > x\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{nx} = e^{-x} \quad \text{for } x\in\mathbb{R}.$$

□

To prevent notation confusion, we let variables w'_{ij} be independent exponentially distributed with mean 1 and the last passage time

$$T'(m,n) = \max_{\pi\in\Pi(m,n)}\left\{\sum_{(i,j)\in\pi} w'_{ij}\right\}.$$

Distribution of $T(m,n)$ in the exponential DLPP

Theorem

For any $m \geq n \geq 1$,

$$\mathbb{P}(T'(m,n) \leq x) = \frac{1}{Z'_{m,n}} \int_{[0,x]^n} \Delta_n(h)^2 \prod_{i=1}^{n} h_i^{m-n} \mathrm{e}^{-h_i} \mathrm{d}\vec{h} \tag{19}$$

where $Z'_{m,n}$ is the normalization constant.

Proof.

$$\mathbb{P}(T'(m,n) \leq x) = \lim_{L\to\infty} \frac{1}{Z_{m,n}} \sum_{0\leq h_i \leq [Lx]+n-1} \Delta_n(h)^2 \prod_{i=1}^{n} \binom{h_i + m - n}{h_i} \left(1 - \frac{1}{L}\right)^{h_i}$$

$$= \lim_{L\to\infty} \frac{L^{(m+1)n}}{Z_{m,n}(m-n)!} \sum_{0\leq h_i \leq [Lx]+n-1} \prod_{1\leq i<j\leq n} \left(\frac{h_i - h_j}{L}\right)^2 \cdot \prod_{i=1}^{n} \mathrm{e}^{-\frac{h_i}{L} + o(\frac{1}{L})} \prod_{k=1}^{m-n} \left(\frac{h_i + k}{L}\right)$$

$$= \frac{1}{Z'_{m,n}} \int_{[0,x]^n} \Delta_n(h)^2 \prod_{i=1}^{n} h_i^{m-n} \mathrm{e}^{-h_i} \mathrm{d}\vec{h}.$$

□

We let $m = [\gamma n]$ and

$$c = (1+\sqrt{\gamma})^2, \ \rho = \frac{(1+\sqrt{\gamma})^{4/3}}{\gamma^{1/6}}.$$

Then by Theorem 4.1,

$$\mathbb{P}(T'([\gamma n], n) \le cn + \rho n^{1/3}x) = \frac{1}{Z'_{[\gamma n],n}} \int_{[0,cn+\rho n^{1/3}x]^n} \Delta_n(h)^2 \prod_{i=1}^{n} h_i^{\alpha_n} e^{-h_i} d\vec{h}$$

where $\alpha_n = (\gamma - 1)n$. this equals the Fredholm determinant

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[x,\infty)^k} \det[\rho n^{1/3} K_n^{\alpha_n}(cn + \rho n^{1/3}x_i, cn + \rho n^{1/3}x_j)]_{i,j=1}^{k} d\vec{x} \tag{20}$$

where

$$K_n^{\alpha}(x,y) = \frac{\kappa_{n-1}}{\kappa_n} \cdot \frac{l_n^{\alpha}(x) l_{n-1}^{\alpha}(y) - l_n^{\alpha}(y) l_{n-1}^{\alpha}(x)}{x-y} \sqrt{x^{\alpha} e^{-x} y^{\alpha} e^{-y}}$$

is the Laguerre kernel, and

$$l_n^{\alpha}(x) = \left(\frac{n!}{(\alpha+n)!}\right)^{1/2} (-1)^n L_n^{\alpha}(x) = \kappa_n x^n + \cdots$$

are the normalized associated Laguerre polynomials,

$$\int_{\mathbb{R}} l_m^{\alpha}(x) l_n^{\alpha}(x) x^{\alpha} e^{-x} dx = \delta_{m,n}.$$

Laguerre kernels

In fact, $L_n^\alpha(x)$ has an explicit formula:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{k+\alpha} x^k \tag{21}$$

By Cauchy integral formula,

$$L_n^\alpha(x) = \frac{\mathrm{e}^x}{2\pi x i} \int_C \frac{\mathrm{e}^{-xz} z^{n+\alpha}}{(z-1)^{n+1}} \mathrm{d}z \tag{22}$$

where C is a circle surrounding $z=1$. From asymptotic formulas for these polynomials it follows that

$$\lim_{n\to\infty} K_n^{\alpha_n}(cn + \rho n^{1/3} x_i, cn + \rho n^{1/3} x_j) = \mathbb{A}(x_i, x_j). \tag{23}$$

This can be proved in the same way as the corresponding results for Meixner polynomials. Using (20), (23) and some estimates we obtain

$$\lim_{n\to\infty} \mathbb{P}(T'([\gamma n], n) \leq cn + \rho n^{1/3} x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[x,\infty)^k} \det[\mathbb{A}(h_i, h_j)]_{i,j=1}^k \mathrm{d}\vec{h} = F_{TW}(x).$$

Thank you!